Group Theory on Quasisymmetry and Protected Near Degeneracy

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In solid state systems, group representation theory is powerful in characterizing the behavior of quasiparticles, notably the energy degeneracy. While conventional group theory is effective in answering *yes-or-no* questions related to symmetry breaking, its application to determining the magnitude of energy splitting resulting from symmetry lowering is limited. Here, we propose a theory on quasisymmetry and near degeneracy, thereby expanding the applicability of group theory to address questions regarding *large-or-small* energy splitting. Defined within the degenerate subspace of an unperturbed Hamiltonian, quasisymmetries form an enlarged symmetry group eliminating the first-order splitting. This framework ensures that the magnitude of splitting arises as a second-order effect of symmetry-lowering perturbations, such as external fields and spin-orbit coupling. We systematically tabulate the quasisymmetry groups within 32 crystallographic point groups and find all the possible unitary quasisymmetry group structures regarding double degeneracy. Applying our theory to the realistic material AgLa, we predict a "quasi-Dirac semimetal" phase characterized by two tiny-gap band anticrossings.

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Introduction.—Symmetry, formulated by group theory, serves as the most basic concept in physics, as it governs the transformation behaviors of wave functions such as selection rules, conserved invariants, and geometric phases. In solid-state systems, the strength of group representation theory applies to the behaviors of quasiparticles, where the degeneracy of energy bands is determined by the dimension of the irreducible representations (irreps) of little groups at certain momenta in the Brillouin zone [1,2]. The recent prosperities of the field of topological phases and topological materials, including exotic quasiparticles [3–26] and novel transport responses [27–37], are based on crystallographic groups, magnetic groups, and spin groups. It is well believed that the power of group representation theory resides in answering the *yes-or-no* questions like if the degeneracy is lifted or if the transition matrix element is zero, according to whether the relevant symmetry is broken. On the other hand, the regime of group theory is hardly employed for addressing the magnitude of energy splitting induced by symmetry lowering, because such *large-or-small* questions are supposed to related to specific characters such as chemical environments and the strength of perturbation. For example, consider a simple tetragonal lattice with space group $P4$ with atomic p_z, d_{z^2}, d_{xy}, and $d_{x^2-y^2}$ orbitals [Fig. 1(a)]. Along the high symmetry line Γ-Z with little group C_4, the two bands, originated from p_z (irrep A) and $d_{x^2-y^2}$ (irrep B), respectively, form an accidental degeneracy when they meet [Fig. 1(c)]; so is the situation for d_{z^2} (irrep A) and d_{xy} (irrep B) orbitals. Both degeneracies are gapped once a strain ϵ_{xy} is introduced reducing the little group to C_2 [Fig. 1(b)], as both two matrix elements $\langle p_z|\epsilon_{xy}|d_{x^2-y^2}\rangle$ and $\langle d_{z^2}|\epsilon_{xy}|d_{xy}\rangle$ transform as the identity representation of C_4. However, conventional representation theory seems to have no prediction on the gap sizes of the two band anticrossings formed by $(p_z, d_{x^2-y^2})$ and (d_{z^2}, d_{xy}).

Indeed, describing or even predicting the magnitude of energy splitting becomes increasingly essential. One notable example is that the tiny gaps along the topological nodal line, caused by spin-orbit coupling (SOC) could lead to large Berry curvature and is thus desirable for anomalous transport phenomena [38–40]. However, such tiny gaps are typically referred to numerical results rather than a more fundamental origin of approximate symmetry. Previous works attempted to evaluate the degree of maintenance of approximate symmetry by introducing fuzzy sets [41,42] or setting artificial thresholds [43–45], or to distinguish distinct topological phases protected by averaged symmetry [46,47]. However, it is unsettled how the magnitude of symmetry-allowed splitting relates to approximate symmetry. Recently, it was proposed that a U(1) symmetry that commutes with the lower-order $\boldsymbol{k}\cdot\boldsymbol{p}$ Hamiltonian exists as

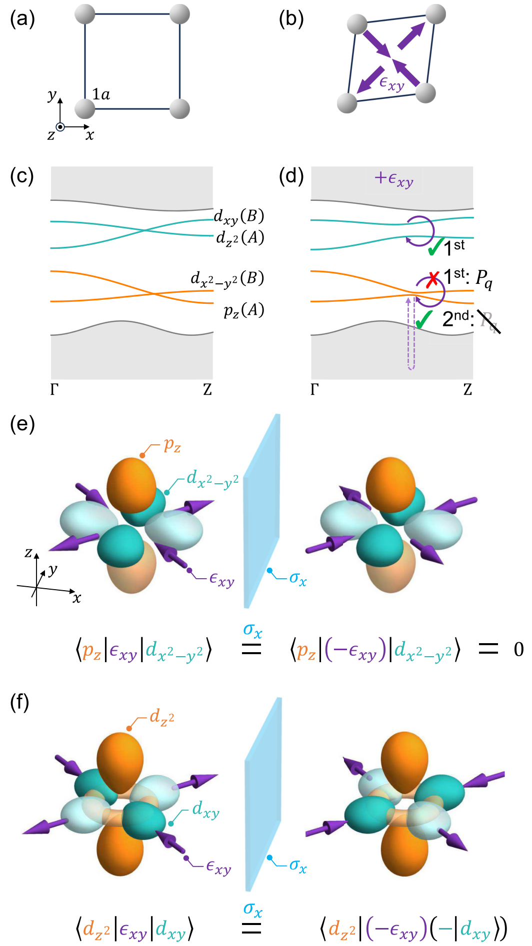

FIG. 1. Schematic of a tetragonal lattice model with space group $P4$. Top view of the tetragonal lattice without (a) and with (b) strain ϵ_{xy}. (c) Accidental band degeneracies along the $\Gamma(0,0,0)$-$Z(0,0,1/2)$ line formed by $(p_z, d_{x^2-y^2})$ and (d_{z^2}, d_{xy}) orbitals settled at $1a$ Wyckoff position, where the corresponding irreps are labeled in parentheses. (d) Strain ϵ_{xy} gaps out both the degeneracies. Protected by quasisymmetry P_q, the gap opened in $(p_z, d_{x^2-y^2})$ bands is a second-order perturbation effect whose size is 1 order smaller than that of the (d_{z^2}, d_{xy}) gap, which is a first-order effect. (e) Quasi-mirror-symmetry σ_x serves as the quasisymmetry to eliminate the first-order effect in $(p_z, d_{x^2-y^2})$ bands. (f) σ_x would not eliminate the first-order effect in (d_{z^2}, d_{xy}) bands.

a so-called quasisymmetry, leading to near degenerate nodal-line structure in a chiral compound CoSi [48,49]. As the concept was conceived in a specific material, a comprehensive and universal symmetry description for near degeneracy (tiny energy splitting) is required for both fundamental understanding of group theory and realistic material design.

In this Letter, we develop a generic theory on quasisymmetry and near degeneracy, and thus expand the application of group representation theory by answering the *large-or-small* question. Defined in the degenerate subspace of an unperturbed Hamiltonian H_0, quasisymmetries form an enlarged symmetry group eliminating the first-order splitting of the symmetry-lowering term H'. Consequently, the magnitude of splitting is ensured to be a second-order effect of the symmetry-lowering perturbation such as external fields and SOC, leading to near degeneracy. We tabulate all the possible quasisymmetry groups within 32 crystallographic point groups, and demonstrate that three types of symmetry groups, i.e., Z_n, U(1), and D_∞ can serve as the quotient group of unitary quasisymmetry group in doubly degenerate subspace. In addition to the tetragonal model where the gap size of $(p_z, d_{x^2-y^2})$ bands is proved to be an order smaller than that of (d_{z^2}, d_{xy}) bands [Fig. 1(d)], we further apply our theory to a realistic material AgLa to predict a SOC-driven phase transition from Dirac nodal-line semimetal to "quasi-Dirac semimetal" exhibiting two tiny-gap band anticrossings. Our work paves a new avenue for designing materials with significant Berry curvature related properties.

Quasisymmetry and elimination of first-order perturbation.—Considering an unperturbed Hamiltonian H_0, all the symmetry operators P_g commuting it form the symmetry group $\mathcal{G}_{H_0} = \{P_g|[P_g, H_0] = 0\}$. Once two eigenstates $|\psi_\alpha\rangle$ and $|\psi_\beta\rangle$, labeled by two inequivalent irreps Γ_α and Γ_β of $\mathcal{G}_{H_0}$, respectively, share the same energy E of H_0, they form an accidental degeneracy [50]. Adding a symmetry-lowering term H' (labeled by irrep Γ_p), the degeneracy splits only if the matrix element $\langle\psi_\alpha|H'|\psi_\beta\rangle$ (labeled by $\Gamma_\alpha^* \otimes \Gamma_p \otimes \Gamma_\beta$) transforms as the identity representation of $\mathcal{G}_{H_0}$, which we termed "$\mathcal{G}_{H_0}$-allowed splitting" for brevity (Supplemental Material, Sec. S1 [51]). As exemplified by the tetragonal lattice model shown in Fig. 1, the conventional group representation theory has no prediction on the magnitude of $\mathcal{G}_{H_0}$-allowed splitting.

Here we demonstrate that near degeneracy, i.e., slightly splitting energy levels induced by H', can also be predicted by symmetry arguments. Such a scenario is realized when the energy splitting is $\mathcal{G}_{H_0}$-allowed but $\langle\psi_\alpha|H'|\psi_\beta\rangle$, known as the first-order perturbation, equals zero, leading to a second-order effect of H'. We next demonstrate that the vanishment of $\langle\psi_\alpha|H'|\psi_\beta\rangle$ is induced by symmetries emerged in eigensubspace of H_0, $\Psi_{\alpha\beta} = \mathrm{Span}(|\psi_\alpha\rangle, |\psi_\beta\rangle)$. Specifically, given a $\mathcal{G}_{H_0}$-allowed splitting, the vanishment of $\langle\psi_\alpha|H'|\psi_\beta\rangle$ can be constrained by a symmetry operator P_q satisfying

$$\langle\psi_\alpha|H'|\psi_\beta\rangle \xrightarrow{P_q} e^{i\omega(P_q)}\langle\psi_\alpha|H'|\psi_\beta\rangle, \qquad \omega(P_q) \bmod 2\pi \neq 0. \tag{1}$$

Note that Eq. (1) implies that P_q is $\Psi_{\alpha\beta}$ invariant, i.e., preserving the eigensubspace $P_q\Psi_{\alpha\beta} = \Psi_{\alpha\beta}$ and $\langle\psi_\alpha|H'|\psi_\beta\rangle = 0$ (Supplemental Material, Sec. S2 [51]). Here the phase ω is generally α, β dependent. The $\Psi_{\alpha\beta}$-invariant symmetry P_q

satisfying Eq. (1) is thus defined as the *quasisymmetry* of eigensubspace $\Psi_{\alpha\beta}$ in H_0, rendering the $\mathcal{G}_{H_0}$-allowed splitting at least a second-order effect with near degeneracy.

Two properties related to quasisymmetry emerge. Firstly, quasisymmetries are excluded from $\mathcal{G}_{H_0}$ because for any $P_g \in \mathcal{G}_{H_0}$, $\omega(P_q)$ mod $2\pi = 0$. Secondly, the concept of quasisymmetry can only be defined for eliminating first-order perturbed Hamiltonian. To prove this, assuming that P_q is a quasisymmetry eliminating the second-order effect $[\propto \sum_{\gamma(\neq\alpha,\beta)} \langle\psi_\alpha|H'|\psi_\gamma\rangle\langle\psi_\gamma|H'|\psi_\beta\rangle(E-E_\gamma)^{-1}]$, we note that P_q must preserves all the eigensubspaces $\Psi_{\alpha\gamma}$ and $\Psi_{\gamma\beta}$ of H_0, yielding that $P_g \in \mathcal{G}_{H_0}$. Thus, it rules out the possibility of "higher-order quasisymmetry." The detailed proof of both properties mentioned above is provided in the Supplemental Material, Sec. S3 [51].

Quasisymmetry group.—Given a symmetry group $\mathcal{G}_{H_0}$ and a symmetry-lowering term H', only certain eigensubspaces $\Psi_{\alpha\beta}$ underpin quasisymmetry. Then two crucial questions arise: which eigensubspaces of $\mathcal{G}_{H_0}$ can support quasisymmetry, and where to look for the corresponding quasisymmetries? Next, we attempt to answer these by extending the exact-symmetry group $\mathcal{G}_{H_0}$ to a so-called quasisymmetry group $\mathcal{Q}(\mathcal{G}_{H_0}, P_q)$, which is hidden inside certain eigensubspace of $\mathcal{G}_{H_0}$. Such extension is based on an important condition on the irreps of quasisymmetry P_q that the $\mathcal{G}_{H_0}$-allowed splitting $\langle\psi_\alpha|H'|\psi_\beta\rangle$ is *not* a $\mathcal{Q}(\mathcal{G}_{H_0}, P_q)$-allowed splitting. Consequently, the matrix element $\langle\psi_\alpha|H'|\psi_\beta\rangle$ transforms as a one-dimensional (1D) nontrivial representation of $\mathcal{Q}(\mathcal{G}_{H_0}, P_q)$, in accordance with Eq. (1). It indicates that the irreps in $\mathcal{G}_{H_0}$ and $\mathcal{Q}(\mathcal{G}_{H_0}, P_q)$ characterizing the eigensubspace $\Psi_{\alpha\beta}$ are highly correlated. Specifically, there must be multiple inequivalent irreps $\Gamma'_{\alpha,1}$ and $\Gamma'_{\alpha,2}$ in $\mathcal{Q}(\mathcal{G}_{H_0}, P_q)$ restricting as the same irrep Γ_α in $\mathcal{G}_{H_0}$ ($\Gamma'_{\alpha,1}\downarrow\mathcal{G}_{H_0} = \Gamma'_{\alpha,2}\downarrow\mathcal{G}_{H_0} = \Gamma_\alpha$), termed as multiple-to-one restrictive condition (Supplemental Material, Sec. S4 [51]).

We now tabulate all the possible eigensubspaces with all the possible quasisymmetry groups in crystallographic point groups. The process is summarized in the following: (i) Starting from a point group $\mathcal{G}_{H_0}$ and a tentative crystallographic symmetry P_q, we construct a quasisymmetry group $\mathcal{Q}(\mathcal{G}_{H_0}, P_q)$ by group extension as

$$1 \to \mathcal{G}_{H_0} \to \mathcal{Q}(\mathcal{G}_{H_0}, P_q) \to \mathcal{F} \to 1, \tag{2}$$

where $\mathcal{F}$ is an Abelian group generated only by P_q. By construction, $\mathcal{G}_{H_0}$ is a normal subgroup of $\mathcal{Q}(\mathcal{G}_{H_0}, P_q)$ and $\mathcal{Q}(\mathcal{G}_{H_0}, P_q)/\mathcal{G}_{H_0} \cong \mathcal{F}$, ensuring at least one irrep in $\mathcal{G}_{H_0}$ is multiple-to-one restrictive (Supplemental Material, Sec. S4 [51]). (ii) We tabulate all the multiple-to-one restrictive irreps in $\mathcal{G}_{H_0}$. Any eigensubspace spanned by these irreps can emerge P_q and other elements in $\mathcal{Q}(\mathcal{G}_{H_0}, P_q)\backslash\mathcal{G}_{H_0}$ as quasisymmetries.

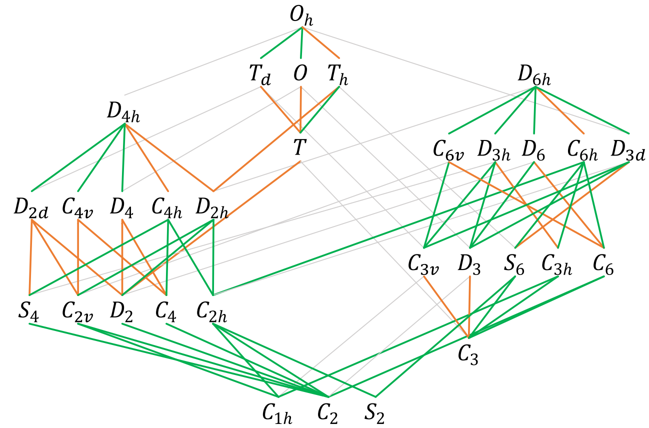

FIG. 2. Quasisymmetry group in 32 crystallographic point groups. A group-subgroup pair is linked by a green (orange) line if all (some of) representations in the subgroup are multiple-to-one restrictive with respect to the parent group. Hence, the parent group could serve as the quasisymmetry group $[\mathcal{Q}(\mathcal{G}_{H_0}, P_q)]$ of the subgroup ($\mathcal{G}_{H_0}$). No quasisymmetry emerges between groups linked by gray lines. Note that point groups C_4 and C_{4v} are involved in the tetragonal lattice model C_{4v} serves as the quasisymmetry group of C_4, while along the T high-symmetry line of AgLa D_{2h} serves as the quasisymmetry group of C_{2v}.

By repeating steps (i) and (ii), all the possible quasisymmetry groups in 32 crystallographic point groups are shown in Fig. 2, and all the multiple-to-one restrictive irreps are tabulated in Tables S1–S5 (Supplemental Material, Sec. S5 [51]). In Fig. 2, a group-subgroup pair is linked by a green (orange) line if the subgroup is normal with all (some of) irreps being multiple-to-one restrictive, indicating that the parent group could be a quasisymmetry group of the subgroup. Interestingly, it is proved that for point groups, $\mathcal{Q}(\mathcal{G}_{H_0}, P_q)$ can always be expressed as a semidirect product $\mathcal{Q}(\mathcal{G}_{H_0}, P_q) = \mathcal{G}_{H_0} \rtimes \mathcal{F}$ or a direct product $\mathcal{G}_{H_0} \times \mathcal{F}$ [2]. Owing to the completeness for crystallographic point groups, Fig. 2 and Tables S1–S5 effectively facilitate the search of the quasisymmetries by referring to the valid group-subgroup pairs even if no P_q is known.

It is worth noting that the constructed quasisymmetry groups could go beyond crystallographic point groups. Taking the double degeneracy, the most practical case, as an example, a unitary quasisymmetry P_q is an element of U(2). We prove that the Abelian quotient group $\mathcal{F}$ generated by $\mathcal{Q}(\mathcal{G}_{H_0}, P_q)$ must be isomorphic to three types of subgroups of U(2), i.e., Z_n, U(1), and D_∞ (Supplemental Material, Sec. S6 [51]). For crystallographic quasisymmetry groups shown in Fig. 2, $\mathcal{F}$ is isomorphic to Z_2 or Z_3. On the other hand, the recent proposed near degenerate nodal line in CoSi belongs to the case with $\mathcal{F} = \mathrm{U}(1)$ [48,49]. Such a Lie group formed by quasisymmetry is also essential for the many-body scar dynamics [63].

Furthermore, the double degeneracy can also contain antiunitary quasisymmetries, which can be constructed by taking the complex conjugate of the eigenstates [64].

Tetragonal lattice model.—To apply our theory, we now present detailed symmetry analysis on the tetragonal lattice model shown in Fig. 1. Conventional representation theory predicts that $\langle p_z|\epsilon_{xy}|d_{x^2-y^2}\rangle$ is a C_4-allowed splitting ($\mathcal{G}_{H_0} = C_4$). We find that mirror reflection σ_x, which is not in $\mathcal{G}_{H_0}$, is $\Psi_{p_z,d_{x^2-y^2}}$ invariant. Moreover, σ_x reverses the strain $\epsilon_{xy} \xrightarrow{\sigma_x} -\epsilon_{xy}$ and transforms $\langle p_z|\epsilon_{xy}|d_{x^2-y^2}\rangle \xrightarrow{\sigma_x} -\langle p_z|\epsilon_{xy}|d_{x^2-y^2}\rangle$ satisfying Eq. (1) [Fig. 1(e)]. Therefore, σ_x is a quasisymmetry of $\Psi_{p_z,d_{x^2-y^2}}$ protecting the degeneracy under the first-order strain effect. Furthermore, the quasireflection σ_x will be broken by involving remote states outside $\Psi_{p_z,d_{x^2-y^2}}$, and the degeneracy will thus be lifted by the second-order effect [Fig. 1(d)] (Supplemental Material, Sec. S7 [51]). In contrast, σ_x is not a quasisymmetry of the eigensubspace spanned by (d_{z^2}, d_{xy}) because $\langle d_{z^2}|\epsilon_{xy}|d_{xy}\rangle \xrightarrow{\sigma_x} \langle d_{z^2}|\epsilon_{xy}|d_{xy}\rangle$, leading to the first-order energy splitting under ϵ_{xy}, as shown in Fig. 1(f).

The inclusion of quasireflection σ_x expands $\mathcal{G}_{H_0} = C_4$ to the quasisymmetry group C_{4v}, with the quotient group $\mathcal{F}$ isomorphic to Z_2. According to our theory, the matrix element $\langle p_z|\epsilon_{xy}|d_{x^2-y^2}\rangle$ (C_4-allowed splitting) is not a C_{4v}-allowed splitting, transforming as $A_1 \otimes B_2 \otimes B_1 = A_2$, a 1D nontrivial representation of C_{4v}. The irreps characterizing $d_{x^2-y^2}$ in C_{4v} and C_4 have the same dimension $\dim B_1 = \dim B$ during the representation restriction $B_1{\downarrow}C_4 = B$. Meanwhile, there is another inequivalent irrep B_2 in C_{4v} restricting as the same irrep B in C_4 ($B_1{\downarrow}C_4 = B_2{\downarrow}C_4 = B$). In turn, by referring to Table S4 it is also straightforward to find that $C_{4v} = C_4 \rtimes S_2$ is a quasisymmetry group of C_4, of which irreps A and B support σ_x as the quasisymmetry.

Application to realistic material AgLa.—We next apply our quasisymmetry group theory to realistic material by taking SOC as the symmetry-lowering perturbation. We choose AgLa (ICSD-58306 [65]) as an example, which has a tetragonal structure with a space group $P4/mmm$ and lattice constants $a = b = 3.656$ Å and $c = 3.840$ Å [Fig. 3(a)]. This compound has been predicted as a topological nodal-line semimetal without SOC by topological quantum chemistry [66]. The combination of inversion and time-reversal symmetry ensures spin degeneracy throughout the Brillouin zone. Our calculations show that without SOC, two spin-degenerate bands intersect around the R point at 0.85–1 eV above the Fermi level [upper panel in Fig. 3(b)], forming a Dirac nodal line (DNL) on the high-symmetry plane $k_y = \pi/a$ [red curve in Fig. 3(c)]. The computational details are shown in the Supplemental Material, Sec. S8 [51].

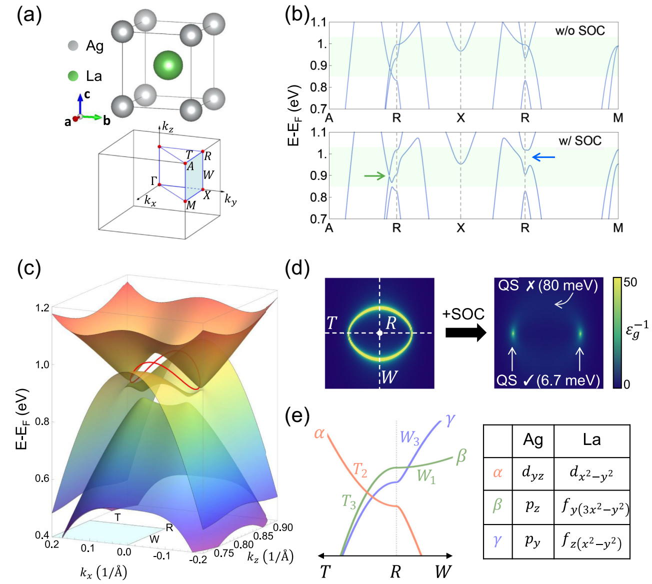

FIG. 3. Quasisymmetry protected tiny gap induced by spin-orbit coupling in AgLa. (a) Crystal structure and Brillouin zone of AgLa. (b) Band structure of AgLa without (upper panel) and with spin-orbit coupling (lower panel) around the high symmetry wave vector R. (c) Visualization of the quasi-Dirac semimetal phase caused by spin-orbit coupling, where the red line marks the Dirac nodal line. (d) Inverse gap ϵ_g^{-1} as a function of k_x (T line) and k_z (W line) with and without spin-orbit coupling, and "QS" denotes quasisymmetry. (e) Orbital projections of the bands α, β, γ forming the nodal line along T and W lines, where the dominant orbitals are tabulated. The crossing points are intersected by representations T_2 (W_3) and T_3 (W_1) along the T (W) line.

When SOC is considered, the entire DNL is gapped, as shown in Figs. 3(b) and 3(c). However, the size of the gap varies dramatically as a function of the wave vector. In the lower panel of Fig. 3(b) we find that the degeneracies crossed at the W (X-R) and F (R-M) lines open ~80 meV gaps (blue arrow), whereas only a 6.7 meV gap is opened at the T (A-R) line (green arrow). We map the inverse gap ϵ_g^{-1} upon the k_x-k_z plane in Fig. 3(d) and observe the minimum of the gap size at the T line. Such interesting distribution of the SOC-induced band gap reveals a novel phase transition from DNL to quasi-Dirac semimetal, the latter of which exhibits gapless Dirac cones under the first-order SOC effect. Such a quasi-Dirac cone is protected by quasisymmetry, and is slightly gapped by the second-order SOC effect. We next show that in AgLa, the quasi-Dirac cones are located at the T line [Fig. 3(c)] and protected by quasisymmetry group D_{2h}.

We obtain the orbital projection of the bands relevant to the DNL along the T and W lines [51], where the d_{yz}, p_z, p_y orbitals from Ag and $d_{x^2-y^2}$, $f_{y(3x^2-y^2)}$, $f_{z(x^2-y^2)}$ from La dominate those bands denoted by α, β, γ in Fig. 3(e), respectively. In the absence of SOC, bands α and β with irreps T_2 and T_3 of the little group $\mathcal{G}_T = C_{2v}$ intersect with each other at the T line, whereas bands β and γ with irreps

W_1 and W_3 of the same little group $\mathcal{G}_W = C_{2v}$ cross at the W line. We next use the theory of quasisymmetry to elucidate the remarkable difference of SOC-induced gap at T and W. By referring to Fig. 2 and Table S5, we find that at wave vector T only D_{2h} can serve as the quasisymmetry group of the eigensubspace spanned by irreps (T_2,T_3) due to the condition of multiple-to-one restrictive irreps, i.e.,

$$\mathcal{Q}(C_{2v}, I) = D_{2h} = C_{2v} \times \{E, I\}. \tag{3}$$

Therefore, inversion I emerges as the candidate of quasisymmetry. We further notice that by taking the on site SOC term $H' = H_{\mathrm{SOC}} \propto \mathbf{L}\cdot\mathbf{S}$, only the z component $\langle\alpha, s|L_z S_z|\beta, s'\rangle$ $(s, s' = \uparrow, \downarrow)$ is $\mathcal{G}_T$-allowed splitting. Such a matrix element in D_{2h} transforms as $A_{2g} \otimes B_{1g} \otimes B_{2u} = A_{1u} \neq A_{1g}$, indicating that D_{2h} is indeed the quasisymmetry group that eliminates the first-order SOC effect. Involving the remote bands breaks the quasi-inversion symmetry and thus opens a second-order SOC gap of 6.7 meV gap. Similarly, for the W line the eigensubspace spanned by irreps (W_1,W_3) also has D_{2h} as the candidate of quasisymmetry group (see Table S5). However, the matrix element in D_{2h} transforms $\langle\beta, s|H_{\mathrm{SOC}}|\gamma, s'\rangle \xrightarrow{I} \langle\beta, s|H_{\mathrm{SOC}}|\gamma, s'\rangle$ with $\omega(I) \bmod 2\pi = 0$. Therefore, I is not a quasisymmetry for the W line according to Eq. (1), leading to a relatively larger gap [51]. Overall, the SOC-driven quasi-Dirac semimetal phase originates from the quasi-inversion emerged only at the L line.

Discussion.—It is worth emphasizing that our theory on quasisymmetry is also valid for the energy splitting of higher-dimensional accidental degeneracy and necessary degeneracy. For the former case, the procedure of analyzing quasisymmetry is the same as that of doubly degenerate band crossings. We show an example of a hexagonal lattice model in the Supplemental Material, Sec. S9 [51]. For the latter case, the symmetry-lowering term H' of irrep Γ_p is supposed to split the degenerate eigensubspace Ψ_α spanned by $(\psi_{\alpha,1}, \psi_{\alpha,2}, \ldots, \psi_{\alpha,N},)$ of N-dimensional irrep Γ_α in $\mathcal{G}_{H_0}$. The matrix elements $\langle\psi_{\alpha,i}|H'|\psi_{\alpha,j}\rangle$ $(i, j = 1, \ldots, N)$ are $\mathcal{G}_{H_0}$-allowed splitting only if $[\Gamma_\alpha \otimes \Gamma_\alpha] \otimes \Gamma_p$ contains the identity representation Γ_1 in $\mathcal{G}_{H_0}$, where $[\Gamma_\alpha \otimes \Gamma_\alpha]$ denotes the symmetric tensor product [5]. The identification of quasisymmetry is the same as that in accidental degeneracy, i.e., Eq. (1). For instance, consider the triplet of $\mathcal{G}_{H_0} = T$ spanned by (p_x, p_y, p_z) (irrep T), where an external electric field $\boldsymbol{\mathcal{E}} = \mathcal{E}_z\hat{z}$ (transform as a partner in irrep T) breaks the symmetry group down to D_2 and hence lifts the triplet due to the condition $[T \otimes T] \otimes T \to A$. By referring to Table S2 we find that irrep T can have T_h as the quasisymmetry group. Furthermore, the matrix element transforms as $\langle p_i|d_z|p_j\rangle \xrightarrow{I} -\langle p_i|d_z|p_j\rangle$ $(i, j = x,\ y,\ z)$ with d_z the electric dipole, resulting in tiny splitting of the triplet protected by quasi-inversion symmetry.

At last, we discuss some possible scenarios and applications for the implementation of quasisymmetry. For example, recent angle-resolved photoelectron spectroscopy measurements have revealed Rashba-like spin splitting with Kramers degeneracy around certain momenta that lack time-reversal symmetry [67], which can be readily explained by the theory of quasisymmetry [68]. More importantly, the key application of quasisymmetry is to generate a substantial anomalous Hall effect by introducing small gaps along the nodal lines in magnetic materials. These small gaps result in significant Berry curvature (Supplemental Material, Sec. S10 [51]), while the extensive distribution of nodal lines enhances the integrated Hall conductivity [38–40,69,70]. Furthermore, it is also possible to create a high-contrast anomalous Hall device sensitive to external field, e.g., a tiny electromagnetic field applied may break quasi-inversion or reflection to create a dip in the Hall signal. Overall, our research paves a new avenue for expanding the scope of group representation theory and designing materials with large Berry curvature and anomalous transport properties.

Note added.—Recently, two experiments have observed the near-quantized double quantum spin Hall effect in twisted bilayer transition-metal dichalcogenide [71,72], indicating that new symmetry indicators of the quantum spin Hall effect are needed. We note that such a phenomenon is protected by the spin U(1) quasisymmetry [73], which is covered by our generic theory in this letter.

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[51] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.133.026402 which includes Refs. [47–57], for notes regarding conventional group theory analysis on energy degeneracy, details of the definitions of quasisymmetry and quasisymmetry group, enumeration of crystallographic quasisymmetry groups and quasisymmetries, quasisymmetries for double degeneracy, analysis of the tetragonal lattice model, and computational details of realistic material AgLa.

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